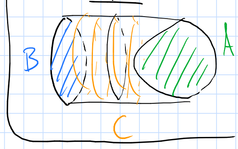
Ex: Compute the flux of $\vec{F} = \langle yz, zx, xy \rangle$ across the outward-oriented boundary of the region enclosed by $x^2+z^2=1$, $y=0$, and $x+y=2$.

Sol 1 (Direct computation): $S = A \cup B \cup C$

Picture



A: $\vec{a}(x,z) = \langle x, 2-x, z \rangle$ on $\{(x,z) : x^2+z^2 \le 1\}$ (↑ $x+y=2$)

$\vec{a}_x = \langle 1, -1, 0 \rangle$
$\vec{a}_z = \langle 0, 0, 1 \rangle$

$\vec{a}_x \times \vec{a}_z = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \langle -1, -1, 0 \rangle$.

↑ inward, so negate!

$\vec{F}(\vec{a}(x,z)) = \langle (2-x)z, xz, x(2-x) \rangle$

$\therefore \vec{F}(\vec{a}(x,z)) \cdot (\vec{a}_x \times \vec{a}_z) = -(2-x)z - xz = -2z + xz - xz = -2z.$

$\therefore \iint_A \vec{F} \cdot d\vec{S} = -\iint_{D_A} \vec{F}(\vec{a}(x,z)) \cdot (\vec{a}_x \times \vec{a}_z)\, dA = -\iint_{D_A} -2z\, dA$ ← use polar $\begin{cases} x = r\cos(\theta) \\ z = r\sin(\theta) \\ dA_{xz} = r\, dA_{r\theta} \end{cases}$

(orientation)

$= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} 2r\sin(\theta) \cdot r\, d\theta\, dr = 2\int_{r=0}^{1} r^2 \int_{\theta=0}^{2\pi} \sin(\theta)\, d\theta\, dr$

$= 2\int_{r=0}^{1} r^2 \left[\cos(\theta)\right]_{\theta=0}^{2\pi} dr = -2\int_{r=0}^{1} r^2 (1-1)\, dr = 0$

B: $\vec{b}(x,z) = \langle x, 0, z \rangle$ on $\{(x,z) : x^2+z^2 \le 1\}$ (↑ $y=0$)

outward ☺

$\vec{b}_x = \langle 1, 0, 0 \rangle = \vec{i}$, $\vec{b}_z = \langle 0, 0, 1 \rangle = \vec{k}$, $\vec{b}_x \times \vec{b}_z = \vec{i} \times \vec{k} = -\vec{j} = \langle 0, -1, 0 \rangle$

$\vec{F}(\vec{b}(x,z)) = \langle 0z, xz, x\cdot 0 \rangle = xz\langle 0, 1, 0 \rangle$.

$\therefore \iint_B \vec{F} \cdot d\vec{S} = \iint_{D_B} \vec{F}(\vec{b}(x,z)) \cdot (\vec{b}_x \times \vec{b}_z)\, dA = \iint_{D_B} -xz\, dA$ ← use polar $\begin{cases} x = r\cos(\theta) \\ y = r\sin(\theta) \\ dA_{xy} = r\, dA_{r\theta} \end{cases}$

$= \int_{r=0}^{1} \int_{\theta=0}^{2\pi} -(r\cos(\theta))(r\sin(\theta))\, r\, d\theta\, dr = -\int_{r=0}^{1} r^3 \int_{\theta=0}^{2\pi} \sin(\theta)\cos(\theta)\, d\theta\, dr$ (trig identity!)

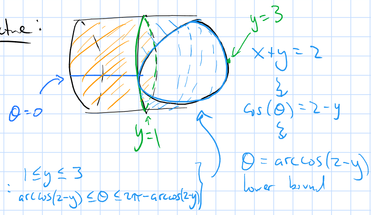
$= -\int_{r=0}^{1} r^3 \int_{\theta=0}^{2\pi} \frac{1}{2}\sin(2\theta)\, d\theta\, dr$

$= -\int_{r=0}^{1} r^3 \frac{1}{4}\left[-\cos(2\theta)\right]_{\theta=0}^{2\pi} dr = \frac{1}{4}\int_{r=0}^{1} r^3 (1-1)\, dr = 0.$

C: $\vec{c}(\theta, y) = \langle \cos(\theta), y, \sin(\theta) \rangle$ ← $x^2+z^2=1$ ← we need to be careful about the bounds now!

$\{(\theta, y) : \begin{matrix} 0 \le \theta \le 2\pi \\ 0 \le y \le 1 \end{matrix}\}$ → orange

Picture:



For the rest we bound θ in y.

$\arccos(2-y) \le \theta \le 2\pi - \arccos(2-y)$.

$\{(\theta, y) : \begin{matrix} 1 \le y \le 3 \\ \arccos(2-y) \le \theta \le 2\pi - \arccos(2-y) \end{matrix}\}$

$\vec{C}_\theta = \langle -\sin(\theta), 0, \cos(\theta) \rangle$, $\vec{C}_y = \langle 0, 1, 0 \rangle$

$\vec{C}_\theta \times \vec{C}_y = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(\theta) & 0 & \cos(\theta) \\ 0 & 1 & 0 \end{bmatrix} = \langle -\cos(\theta), 0, -\sin(\theta) \rangle$

↑ oriented inward (e.g. at $\theta = 0$). So negate!

$\vec{F}(\vec{c}(\theta, y)) = \langle y\sin(\theta), \sin(\theta)\cos(\theta), y\cos(\theta) \rangle$

$\therefore \vec{F}(\vec{c}(\theta, y)) \cdot (\vec{C}_\theta \times \vec{C}_y) = -y\cos(\theta)\sin(\theta) + 0 - y\cos(\theta)\sin(\theta)$

$= -2y\sin(\theta)\cos(\theta) = -y\sin(2\theta)$ ← trig identity

$\therefore \iint_C \vec{F} \cdot d\vec{S} = -\iint_{D_C} -y\sin(2\theta)\, dA$ (from orientation)

$= \int_{y=0}^{1} \int_{\theta=0}^{2\pi} y\sin(2\theta)\, d\theta\, dy + \int_{y=1}^{3} \int_{\theta=\arccos(2-y)}^{2\pi-\arccos(2-y)} y\sin(2\theta)\, d\theta\, dy$

$\int_{\theta=0}^{2\pi} \sin(2\theta)\, d\theta = 0$ from earlier

$= \int_{y=1}^{3} y \int_{\theta=\arccos(2-y)}^{2\pi-\arccos(2-y)} \sin(2\theta)\, d\theta\, dy = -\frac{1}{2}\int_{y=1}^{3} \left[\cos(2\theta)\right]_{\theta=\arccos(2-y)}^{2\pi-\arccos(2-y)} dy$

We'll simplify before continuing:

$\left[\cos(2\theta)\right]_{\theta=\arccos(2-y)}^{2\pi-\arccos(2-y)} = \cos\left(4\pi - 2\arccos(2-y)\right) - \cos\left(2\arccos(2-y)\right)$

$= \cos(4\pi)\cos(2\arccos(2-y)) + \sin(4\pi)\sin(2\arccos(2-y)) - \cos(2\arccos(2-y))$

($\cos(4\pi) = 1$, $\sin(4\pi) = 0$)

($\cos(\alpha-\beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$)

$= \cos(2\arccos(2-y)) - \cos(2\arccos(2-y)) = 0$ ↑ nice ☺

$\therefore \iint_C \vec{F} \cdot d\vec{S} = \int_{y=1}^{3} 0\, dy = 0$

$\therefore \iint_S \vec{F} \cdot d\vec{S} = \iint_A \vec{F} \cdot d\vec{S} + \iint_B \vec{F} \cdot d\vec{S} + \iint_C \vec{F} \cdot d\vec{S} = 0 + 0 + 0 = 0$ ▨

Sol 2 (Divergence Theorem): $S = \partial R$ for the region described in the problem. Thus we can apply Divergence Theorem:

$\operatorname{div}(\vec{F}) = \nabla \cdot \vec{F} = \frac{\partial}{\partial x}[yz] + \frac{\partial}{\partial y}[xz] + \frac{\partial}{\partial z}[xy] = 0$

$\iint_{\partial R} \vec{F} \cdot d\vec{S} = \iiint_R \operatorname{div}(\vec{F})\, dV = \iiint_R 0\, dV = 0$ ☺ ▨